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EXTENSION AND APPLICATION OF
A SEQUENTIAL ESTIMATOR

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ABSTRACT

This thesis extends and applies a particular sequential estimation technique. First a method is presented to increase the accuracy of the estimator, given bounds on the measurement noise. Second a method is presented where some systems may be estimated adequately with a greatly simplified estimator which results in a large reduction in the amount of computation required by the estimator. Examples are discussed showing how these methods have been applied successfully.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF GRAPHS

LIST OF MAJOR SYMBOLS

A	$n \times n$ constant system matrix
B	$n \times n$ constant matrix, diagonal, positive definite
b_i	ith diagonal element of B
C	n-vector variable
$e_1(t)$	n-vector, residual error
$e_2(t)$	n-vector, residual error
$\tilde{e}(t)$	n-vector, error in estimation
$\tilde{e}_o(t)$	n-vector, estimation error of original estimator
$\tilde{e}_{oi}(t)$	scalar, ith component of $\tilde{e}_o(t)$
$\tilde{e}_m(t)$	n-vector, estimation error of extended estimator
$\tilde{e}_{mi}(t)$	scalar, ith component of $\tilde{e}_m(t)$
$f(t, x)$	n-vector function, equation (1-9)
$f'(t, x)$	n-vector function, equation (3-5)
$g_o(t, x)$	n-vector function, completely known
$g_{o\tilde{x}}$	n-vector, equation (1-27)
$H(t_f, \hat{x})$	$1 \times n$-vector, equation (1-27)
$H(t, \overline{x}, \lambda, w)$	scalar Hamiltonian function
$h(t, x)$	d-vector function, known
$h_i(t)$	scalar function, equation (2-18)
$J1$	scalar cost function, equation (1-4)
$J2$	scalar cost function, equation (2-1)
$J2_o$	scalar cost function, equation (2-5)
$J2_m$	scalar cost function, equation (2-7)

$k(t, x)$	$n \times q$-vector function, unknown
$M(t)$	scalar function, equation (2-25)
n	scalar, order of system
$P(t)$	$n \times n$-matrix of matrix Riccati equation
P_{ss}	$n \times n$-matrix, steady state solution for $P(t)$
Q	$n \times n$ scalar matrix
$r(C, t_f)$	n-vector function, equation (1-20)
$r_1(t)$	scalar random variable
$r_2(t)$	scalar random variable
$s(t, x)$	n-vector function
t	scalar, time
t_f	scalar, final time
$u(t)$	q-vector unknown input
$v(t)$	d-vector observation errors
v_m	d-vector, maximum bounds on $v(t)$ components
W	$n \times n$ scalar matrix
$w(t, \overline{x}, y)$	n-vector input to estimator
w^*	n-vector, optimum value of $w(t, \overline{x}, y)$
$x(t)$	n-vector, system state
$\dot{x}(t)$	n-vector, derivative of $x(t)$
$\overline{x}(t)$	n-vector, estimate of $x(t)$
$\hat{x}(t)$	n-vector, optimum estimate of $x(t)$
$\hat{x}_o(t)$	n-vector, original estimate of $x(t)$
$\hat{x}_m(t)$	n-vector, extended estimate of $x(t)$
x_i	scalar, ith element of $x(t)$
$\hat{x}_{oi}$	scalar, ith element of $\hat{x}_o(t)$

$\hat{x}_{mi}$	scalar, $i\underline{th}$ element of $\hat{x}_m(t)$
$y(t)$	d-vector output of system
$\Delta\tilde{e}_o(t)$	n-vector, equation (2-4)
$\Delta\tilde{e}_{oi}$	scalar, $i\underline{th}$ element of $\Delta\tilde{e}_o(t)$
$\Delta g(t, x)$	n-vector function, unknown
λ	n-vector, Lagrange multiplier
λ^T	$1 \times$ n-vector, transpose of λ
W^{-1}	$n \times n$ matrix, inverse of W
$\|\|\cdot\|\|_Q$	scalar, generalized norm, equation (1-5)

CHAPTER I

INTRODUCTION

The purpose of this report is to extend and apply the sequential estimation technique reported in [1], expanded in [2] and [3], and utilized in [4], [5] and [6]. The thesis is divided into two major parts. Chapter II deals with a method of improving state estimation for non-linear time varying systems. Chapter III deals with the practical application of the above techniques and shows how the amount of computation required by the estimator may be greatly reduced.

A. Purpose

The extension of the estimator improves estimation in a large class of non-linear time varying systems. These systems are those which have a scalar output which is a linear combination of the system states. This output is corrupted by additive measurement noise. The technique is based on the assumption that this noise has a known bound on its amplitude. The improvement in estimation is with respect to a performance index which is the integral of the sum of the squares of the weighted estimation errors. This index is reduced by generating an improved estimate of at least one of the system states using the output of the original estimator and knowledge of the bound on the measurement noise. The advantage of this technique is that it improves estimation and can be implemented easily. In some cases it also assures convergence of some of the estimation errors to finite bounds.

The second part of the report deals with the practical application of the estimator discussed in [3]. In high order systems the amount of computation required by the estimator becomes extremely large so that it is difficult to implement the estimator in real time. For some non-linear and/or time varying systems it is possible to "linearize" the estimator in such a way that adequate if not optimal estimation occurs. This is done by solving the matrix Riccati equation required by the estimator for its steady state values by first removing the non-linear and time varying terms from the system. These steady state values are then used in the remaining estimator equations rather than those values which would have come from the time varying solution of the matrix Riccati equation. This greatly reduces the computation required by the estimator because the matrix Riccati equation is only solved once and this may be done "off-line." The disadvantage of this method is that the loss in estimation accuracy may vary greatly from system to system and that some systems cannot be formulated in this manner.

B. Summary of the Estimation Technique

Before proceeding, the basic estimation technique as developed in [3] is summarized. The problem considered is that of obtaining an optimum estimate of the state of systems described by vector differential equations of the form

$$\dot{x}(t) = g_0(t, x) + \Delta g(t, x) + k(t, x)u(t) \tag{1-1}$$

$$y(t) = h(t, x) + v(t) \tag{1-2}$$

where x, an n-vector, is the system state vector; $g_0(t, x)$ is an n-vector function those variation with t and x is completely known; $\Delta g(t, x)$ and $k(t, x)$ are n-vector and $n \times q$ vector functions, respectively, whose variations with t are unknown; u(t) is a q-vector unknown input which is undefined in a statistical sense; h(t, x) is a known d-vector function; y(t) is a d-vector output; and v is a d-vector of observation errors which are undefined in a statistical sense. The estimator is chosen to satisfy the equation

$$\dot{\overline{x}}(t) = g_0(t, \overline{x}) + w(t, \overline{x}, y) \tag{1-3}$$

where $\overline{x}$, an n-vector, is any estimate of the state vector x, and $w(t, \overline{x}, y)$ is an n-vector input to the estimator which is to be determined.

The optimum estimate of x(t), denoted by $\overline{x}(t) = \hat{x}(t)$, will be optimum in a least squares sense. That is, the estimation problem solved is the following: given output measurements y(t) in the interval $0 \leq t \leq t_f$, determine an estimate $\hat{x}(t_f)$, of the current state vector $x(t_f)$, which is optimum in the sense that it minimizes the functional

$$J1 = \int_0^{t_f} \left[||e_1(t)||_Q^2 + ||e_2(t)||_W^2 \right] dt \tag{1-4}$$

where $||\cdot||_Q$ and $||\cdot||_W$ are generalized norms defined by

$$||e_1(t)||_Q^2 = e_1(t)^T Q e_1(t) \tag{1-5}$$

$$||e_2(t)||_W^2 = e_2(t)^T W e_2(t) \tag{1-6}$$

where Q and W are n × n positive definite matrices and the superscript T denotes transpose. The following relations pertaining to (1-4) also hold:

$$e_1(t) = y(t) - h(t, \overline{x}) \tag{1-7}$$

$$e_2(t) = f(t, \overline{x}) - w(t, \overline{x}, y) \tag{1-8}$$

$$f(t, \overline{x}) = \Delta g(t, \overline{x}) + k(t, \overline{x})u(t) \tag{1-9}$$

If equations (1-7), (1-8) and (1-9) are substituted into (1-4) the result is:

$$J1 = \int_0^{t_f} \left[||y - h(t, \overline{x})||_Q^2 + ||f(t, \overline{x}) - w(t, \overline{x}, y)||_W^2 \right] dt \tag{1-10}$$

The problem now is that of minimizing (1-10) with respect to $w(t, \overline{x}, y)$, subject to the constraint equation (1-3). It is noted that minimizing the integral of the residual errors, i.e., $e_1(t)$ and $e_2(t)$, squared is not the same as minimizing the integral of the estimation errors squared, where the estimation error is an n-vector defined by:

$$\tilde{e}(t) = x(t) - \hat{x}(t) \tag{1-11}$$

Hence, this leaves room for improvement in the estimator after the optimization process has been carried out.

The solution of this optimization problem is now summarized.

Pontryagin's maximum principle is employed in this solution. Toward this end, the Hamiltonian $H(t, \overline{x}, \lambda, w)$ is defined as

$$H(t, \overline{x}, \lambda, w) = ||y - h(t, \overline{x})||_Q^2 + ||f(t, \overline{x}) - w(t, \overline{x}, y)||_W^2 + \lambda^T(g_o(t, \overline{x}) + w(t, \overline{x}, y)) \tag{1-12}$$

where λ, the Lagrange multiplier, is an n-vector.

The Hamiltonian is minimized with respect to $w(t, \overline{x}, y)$ by setting

$$\frac{\partial H}{\partial w} \equiv 0 . \tag{1-13}$$

The solution of (1-13) yields

$$w^*(t, \overline{x}, y) = f(t, \overline{x}) - \frac{1}{2} W^{-1}\lambda \tag{1-14}$$

where w^* is the optimum estimator input and W^{-1} is the inverse of W. The use of (1-14) in (1-12) leads to

$$H^*(t, \overline{x}, \lambda) = ||y - h(t, \overline{x})||_Q^2 + \lambda^T g_o(t, \overline{x}) + \lambda^T f(t, \overline{x}) - \frac{1}{4} \lambda^T W^{-1} \lambda \tag{1-15}$$

where $\overline{x}(t)$ is the estimate corresponding to $w^*(t, \overline{x}, y)$.

It is necessary that $\overline{x}$ and λ satisfy the canonical equations:

$$\dot{\overline{x}} = \frac{\partial H^*}{\partial \lambda}(t, \overline{x}, \lambda) \qquad \dot{\lambda} = -\frac{\partial H^*}{\partial \overline{x}}(t, \overline{x}, \lambda) \tag{1-16}$$

The terminal time t_f has been fixed, and $\overline{x}(0)$ and $\overline{x}(t_f)$ are free. Thus, transversality requires that:

$$\lambda(0) = 0 \qquad \lambda(t_f) = 0 \tag{1-17}$$

If the two point boundary value problem (hereafter referred to as TPBVP) represented by (1-16) and (1-17) is solved, it yields $\hat{x}(t_f)$, i.e., the least squares estimate of $x(t_f)$. For a time greater than t_f, say t_{f1}, the TPBVP must be solved again using the boundary conditions:

$$\lambda(0) = 0 \qquad \lambda(t_{f1}) = 0 \tag{1-18}$$

In order to avoid solving a different TPBVP for each value of final time, the problem is recast as a sequential problem in which t_f is an independent variable, the running time variable. Thus, this TPBVP is embedded in a larger class of TPBVP's. Let this TPBVP be embedded into the larger class of TPBVP's with boundary conditions:

$$\lambda(0) = 0 \qquad \lambda(t_f) = C \tag{1-19}$$

The missing terminal condition on $\overline{x}$ is denoted by

$$\overline{x}(t_f) = r(C, t_f) \tag{1-20}$$

where C and t_f are regarded as independent variables. From (1-16) and (1-19) it can be shown [1] that $r(C, t_f)$ must satisfy

$$\frac{\partial r}{\partial t_f} - \frac{\partial r}{\partial C} \cdot \frac{\partial H^*}{\partial r}(t_f, r, C) = \frac{\partial H^*}{\partial C}(t_f, r, C) \tag{1-21}$$

where

$$\frac{\partial r}{\partial C} = \left[\frac{\partial r_i}{\partial C_j}\right]_{n\times n} \tag{1-22}$$

$$\frac{\partial r}{\partial t_f} = \left[\frac{\partial r_i}{\partial t_f}\right]_{n\times 1} \tag{1-23}$$

$$\frac{\partial H^*}{\partial r} = \left[\frac{\partial H^*}{\partial r_i}\right]_{n\times 1} \qquad \frac{\partial H^*}{\partial C} = \left[\frac{\partial H^*}{\partial C_i}\right]_{n\times 1} \tag{1-24}$$

Equation (1-15) is substituted into (1-21) and an approximate solution of the form

$$r(C,\ t_f) = P(t_f)C + \hat{x}(t_f) \tag{1-25}$$

is tried. Expanding the result about $r(0,\ t_f)$ and noting that only those solutions for $C = 0$ are of interest, the sequential estimator equations become:

$$\frac{d\hat{x}}{dt_f} = g_o(t,\ \hat{x}) + 2P(t_f)H(t_f,\ \hat{x})Q\{y(t_f) - h(t_f,\ \hat{x})\} + f(t_f,\ \hat{x}) \tag{1-26}$$

$$\frac{dP}{dt_f} = g_{o\hat{x}}(t_f,\ \hat{x})P(t_f) + P(t_f)g^T_{o\hat{x}}(t_f,\ \hat{x}) + 2P(t_f)[HQ\{y(t_f) - h(t_f,\ \hat{x})\}]_{\hat{x}}P(t) + \frac{1}{2}W^{-1} \tag{1-27}$$

where

$$g_{o\hat{x}} = \frac{\partial g_o}{\partial \hat{x}}(t_f,\ \hat{x}), \qquad H(t_f,\ \hat{x}) = \left[\frac{\partial h(t_f,\ \hat{x})}{\partial \hat{x}}\right]^T$$

and

$$[HQ\{y(t_f) - h(t_f,\ \hat{x})\}]_{\hat{x}}$$

is an $n \times n$ matrix with ith column

$$\frac{\partial}{\partial \hat{x}_i}[HQ\{y(t_f) - h(t_f,\ \hat{x})\}] .$$

Note that equation (1-27) is known as the matrix Riccati equation.

CHAPTER II

EXTENDED ESTIMATOR

A. Problem Definition

The estimator summarized in Chapter I is optimum in the sense that it is designed to minimize the cost function J1 defined in (1-4). This cost function depends only indirectly upon the error in estimation defined in (1-11). It would be much better if an estimator could be designed on the basis of minimizing the functional

$$J2 = \int_0^{t_f} ||\tilde{e}(t)||_B^2 \, dt \tag{2-1}$$

where $\tilde{e}(t)$ is the error in estimation. This is a difficult problem which has not been solved for the class of systems under consideration. However, the estimator developed in Chapter I has associated with it a cost J2 for each final time, t_f. The problem considered in this chapter is that of generating a new set of estimates of $x(t)$ based on the original estimate $\hat{x}(t)$ and on the bounds on the measurement noise which are assumed to be known. The new estimates will reduce the value of J2 for each t_f from that value generated by the original estimates. Hence, in a practical sense the estimation has been improved. The value of the original cost function J1 may be increased, but since J2 is a better measure of the accuracy in estimation an increase in J1 is acceptable.

B. Derivation of Extended Estimator

It is convenient to begin the derivation with a set of definitions. Define:

$x(t)$ - $n \times 1$ system state vector

$\hat{x}_o(t)$ - $n \times 1$ estimate of $x(t)$ generated by the estimator summarized in Chapter I, Section B.

$\hat{x}_m(t)$ - $n \times 1$ improved estimate of $x(t)$ to be derived in this section.

$\tilde{e}_o(t)$ - error in estimation

$$\tilde{e}_o(t) = x(t) - \hat{x}_o(t) \tag{2-2}$$

$\tilde{e}_m(t)$ - reduced error in estimation

$$\tilde{e}_m(t) = x(t) - \hat{x}_m(t) \tag{2-3}$$

$\Delta\tilde{e}_o(t)$ - incremental change in estimation error

$$\Delta\tilde{e}_o(t) = \tilde{e}_m(t) - \tilde{e}_o(t) \tag{2-4}$$

$J2_o$ - performance index for $\hat{x}_o(t)$

$$J2_o = \int_0^{t_f} ||\tilde{e}_o(t)||_B^2 \, dt \tag{2-5}$$

$$||\tilde{e}_o(t)||_B^2 = \tilde{e}_o(t)^T B \tilde{e}_o(t) \tag{2-6}$$

$J2_m$ - performance index for $\hat{x}_m(t)$

$$J2_m = \int_0^{t_f} ||\tilde{e}_m(t)||_B^2 \, dt \tag{2-7}$$

$$||\tilde{e}_m(t)||_B^2 = \tilde{e}_m(t)^T B \tilde{e}_m(t) \tag{2-8}$$

To obtain a solution to this problem it is necessary to restrict the $n \times n$ positive definite matrix B to be diagonal. This makes the derivation which follows possible. The restriction is reasonable because it makes the performance index equal to the integral of the sum of the weighted squares of the estimation errors. This is a direct and clear-cut measure of the accuracy of the estimation.

The object is now to find some estimate, $\hat{x}_m(t)$, which insures that for all $t_f \geq 0$

$$J2_m - J2_o \leq 0 \ . \tag{2-9}$$

If this condition holds, then the new estimate, $\hat{x}_m(t)$, is an improvement over $\hat{x}_o(t)$ with respect to the chosen performance index. The choice of $\hat{x}_m(t)$ is based on the following:

(a) $\hat{x}_o(t)$ - the original estimate

(b) $y(t)$ - the measurements of the system

(c) $y(t) = h(t, x) + v(t)$ - equation (1-2)

(d) $h(t, x)$ - known function, linear in $x(t)$

(e) v_m - bound on magnitude of $v(t)$

For simplicity, this derivation assumes that $y(t)$ and $v(t)$ are scalars. Hence the bound on $v(t)$ is v_m, i.e.,

$$|v(t)| \leq v_m \text{ for all } t \ . \tag{2-10}$$

Using the above information $\hat{x}_m(t)$ will now be derived such that the inequality (2-9) is satisfied for all $t_f \geq 0$.

The derivation begins by using (2-5), (2-6), (2-7) and (2-8) to get

$$J2_m - J2_o = \int_0^{t_f} \tilde{e}_m^T B\tilde{e}_m \, dt - \int_0^{t_f} \tilde{e}_o^T B\tilde{e}_o \, dt$$

or

$$J2_m - J2_o = \int_0^{t_f} \left[\tilde{e}_m^T B\tilde{e}_m - \tilde{e}_o^T B\tilde{e}_o\right] dt \ .$$

Using (2-4), i.e., $\tilde{e}_m = \tilde{e}_o + \Delta\tilde{e}_o$ in the above equation gives

$$J2_m - J2_o = \int_0^{t_f} [(\tilde{e}_o + \Delta\tilde{e}_o)^T B(\tilde{e}_o + \Delta\tilde{e}_o) - \tilde{e}_o^T B\tilde{e}_o]\ dt$$

$$= \int_0^{t_f} [2\Delta\tilde{e}_o^T B\tilde{e}_o + \Delta\tilde{e}_o^T B\Delta\tilde{e}_o]\ dt$$

$$J2_m - J2_o = \int_0^{t_f} \Delta\tilde{e}_o^T B[2\tilde{e}_o + \Delta\tilde{e}_o]\ dt \qquad (2\text{-}11)$$

Suppose a $\Delta\tilde{e}_o$ can be found such that $J2_m - J2_o \leq 0$ for all $t_f \geq 0$ and all $\tilde{e}_o$, then from (2-2), (2-3) and (2-4)

$$\Delta\tilde{e}_o = \tilde{e}_m - \tilde{e}_o = (x - \hat{x}_m) - (x - \hat{x}_o)$$

$$\Delta\tilde{e}_o = -\hat{x}_m + \hat{x}_o$$

or

$$\hat{x}_m = \hat{x}_o - \Delta\tilde{e}_o. \qquad (2\text{-}12)$$

The estimate $\hat{x}_o$ is known so the selection of $\Delta\tilde{e}_o$ is equivalent to the selection of $\hat{x}_m$, the improved estimate, and condition (2-9) will be satisfied.

Summarizing, pick $\Delta\tilde{e}_o$ such that

$$J2_m - J2_o = \int_0^{t_f} \Delta\tilde{e}_o^T\ B(\Delta\tilde{e}_o + 2\tilde{e}_o)\ dt \leq 0 \qquad (2\text{-}13)$$

for all $t_f \geq 0$ and all $\tilde{e}_o$. When this problem is solved, the result is that there still remains some freedom in the choice of $\Delta\tilde{e}_o$. This freedom is used to minimize $J2_m - J2_o$ so that the new estimate will be the greatest possible improvement over $\hat{x}_o$. The solution of these problems follows.

Since (2-13) must hold for all t_f and all $\tilde{e}_o$, it is sufficient that the integrand be nonpositive, i.e.,

$$\Delta\tilde{e}_o^T B(\Delta\tilde{e}_o + 2\tilde{e}_o) \leq 0$$

or

$$\Delta\tilde{e}_o^T B\Delta\tilde{e}_o + 2\Delta\tilde{e}_o^T B\tilde{e}_o \leq 0 \ . \qquad (2\text{-}14)$$

Define

$$\Delta\tilde{e}_o = \begin{bmatrix} \Delta\tilde{e}_{o1} \\ \Delta\tilde{e}_{o2} \\ \cdot \\ \cdot \\ \cdot \\ \Delta\tilde{e}_{on} \end{bmatrix} \qquad \tilde{e}_o = \begin{bmatrix} \tilde{e}_{o1} \\ \tilde{e}_{o2} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{e}_{on} \end{bmatrix} \qquad (2\text{-}15)$$

and

$$B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \qquad (2\text{-}16)$$

where $b_j > 0$ for $i = 1, 2, \ldots, n$ because B is positive definite.

Expanding (2-14) using (2-15) and (2-16) gives

$$\Delta\tilde{e}_{o1}^2 b_1 + \Delta\tilde{e}_{o2}^2 b_2 + \cdots + \Delta\tilde{e}_{on}^2 b_n + 2\Delta\tilde{e}_{o1}\tilde{e}_{o1}b_1 +$$
$$+ 2\Delta\tilde{e}_{o2}\tilde{e}_{o2}b_2 + \cdots + 2\Delta\tilde{e}_{on}\tilde{e}_{on}b_n \leq 0. \qquad (2\text{-}17)$$

Before going on with this result it is necessary to work with $h(t, x)$. Since this function is assumed to be linear in $x(t)$ it may be written as

$$h(t, x) = h_1(t)x_1 + h_2(t)x_2 + \cdots + h_n(t)x_n \qquad (2\text{-}18)$$

where x_i is the i<u>th</u> component of $x(t)$ and the $h_i(t)$ for $i = 1, 2, \ldots, n$

are known functions. With the object of satisfying (2-9) in mind, pick $\Delta\tilde{e}_o$ in the following manner. Let

$$\Delta\tilde{e}_{oi} = (h_i(t)/b_i)\alpha \tag{2-19}$$

for $i = 1, 2, \ldots, n$ where the value of α remains to be selected. Then using (2-19) in (2-17) and simplifying gives

$$\alpha^2 \sum_{i=1}^{n} h_i(t)^2/b_i + 2\alpha \sum_{i=1}^{n} h_i(t)\tilde{e}_{oi} \leq 0. \tag{2-20}$$

Noting however that

$$h(t, \tilde{e}_o) = \sum_{i=1}^{n} h_i(t)\tilde{e}_{oi} \tag{2-21}$$

then (2-20) becomes

$$\alpha^2 \sum_{i=1}^{n} h_i(t)^2/b_i + 2\alpha h(t, \tilde{e}_o) \leq 0. \tag{2-22}$$

Equation (2-2) and the linearity of $h(t, x)$ gives

$$h(t, \tilde{e}_o) = h(t, (x - \hat{x}_o)) = h(t, x) - h(t, \hat{x}_o). \tag{2-23}$$

Using (1-2) in (2-23)

$$h(t, \tilde{e}_o) = y(t) - v(t) - h(t, \hat{x}_o)$$

and with this result (2-22) becomes

$$\alpha^2 \sum_{i=1}^{n} h_i(t)^2/b_i + 2\alpha(y - h(t, \hat{x}_o) - v(t)) \leq 0. \tag{2-24}$$

For simplicity define

$$M(t) = \sum_{i=1}^{n} h_i^2(t)/b_i \;. \tag{2-25}$$

Note that $M(t) \geq 0$ because each b_i is positive. Define also

$$\gamma = y - h(t, \hat{x}_o) \ . \tag{2-26}$$

Simplifying (2-24) using (2-25) and (2-26) gives

$$\alpha^2 M(t) + 2\gamma\alpha - 2v\alpha \leq 0 \ . \tag{2-27}$$

The problem is now to pick a value for α so that (2-27) holds for all $v(t)$, $|v| \leq v_m$, given known $M(t)$ and γ. Then (2-9) will be satisfied and $\Delta\tilde{e}_o$ and $\hat{x}_m$ can be generated.

This will be done by finding a region R in the (γ, α) plane where (2-27) holds for all $v(t)$ such that $|v| \leq v_m$. This region R will be a function of $M(t)$.

Suppose $\alpha > 0$, then (2-27) becomes

$$\alpha M + 2\gamma - 2v \leq 0 \ . \tag{2-28}$$

Examine this inequality for the extremes of v.

(a) If $v = v_m$, then $\alpha \leq -(2/M)(\gamma - v_m)$. Since $\alpha > 0$ and $M > 0$, this inequality is valid if and only if $\gamma < v_m$.

(b) If $v = 0$, then $\alpha \leq -(2/M)(\gamma)$ for $\gamma < 0$.

(c) If $v = -v_m$, then $\alpha \leq -(2/M)(\gamma + v_m)$ for $\gamma < -v_m$.

Since v may range anywhere within its bounds, the part of the R region for $\alpha > 0$ is that region of the (γ, α) plane where conditions a, b and c above hold simultaneously. A directly analogous argument holds for the part of R when $\alpha < 0$. See Figure 1 for a picture of the region R derived above.

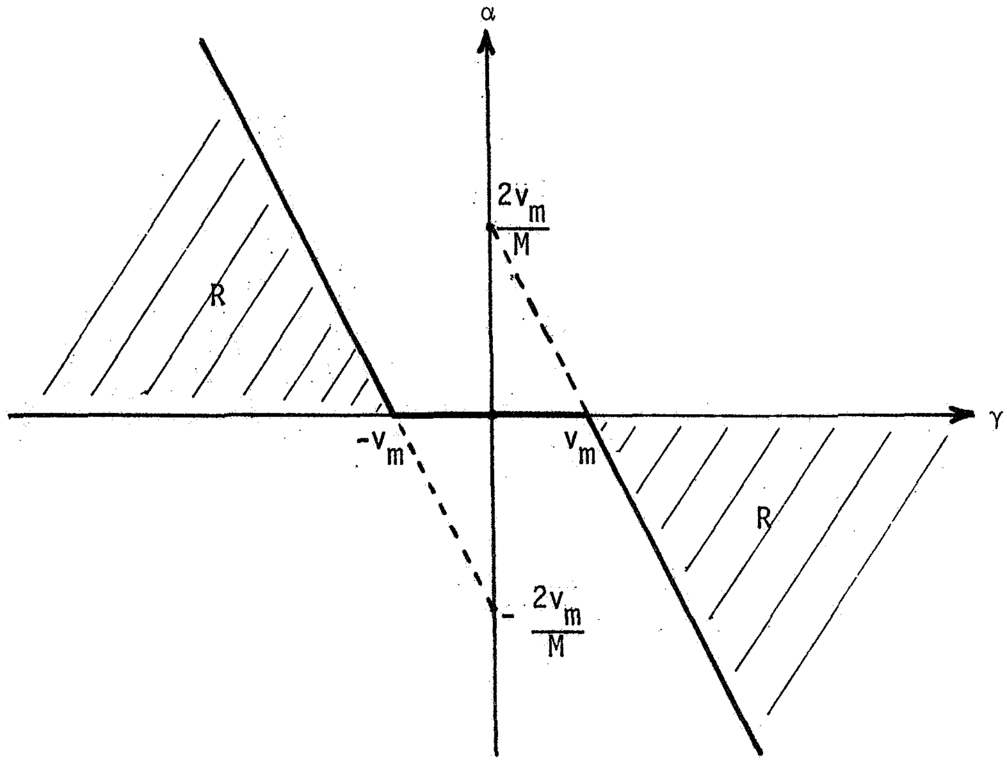


FIGURE 1. Region R

For any given γ a value of α may be chosen in R and (2-9) will be satisfied as desired. If $|\gamma| < v_m$, then α is picked equal to zero and (2-9) is still satisfied. Hence, it is assured that with the above selection of α, which determines $\Delta\tilde{e}_o$ and finally $\hat{x}_m$, the difference $J2_m - J2_o$ will be negative or zero for all t_f and all $\tilde{e}_o$. It is now a matter of trying to pick the best value of α in R so that $J2_m - J2_o$ will be as negative as possible and therefore give the largest improvement in estimation.

Repeating equation (2-27) and setting the left side equal to β gives

$$\beta = \alpha^2 M + 2\gamma\alpha - 2v\alpha \leq 0 \ .$$

Now minimize β with respect to α in the region R by setting

$$\frac{\partial\beta}{\partial\alpha} \equiv 0$$

$$\frac{\partial\beta}{\partial\alpha} \equiv 0 = 2\alpha M + 2\gamma - 2v\ .$$

So,

$$\alpha = -(\gamma - v)/M(t) \text{ for } (\gamma, \alpha) \text{ in } R. \tag{2-29}$$

Choosing α using equation (2-29) requires the knowledge of v to optimize the choice, but v is totally unknown except for its bounds. If the mean of v were known it would be best to use its value in equation (2-29), but since this is not known in general, any value of v within its bounds will do. Pick v = 0 in (2-29) as the most convenient. Then (2-29) becomes

$$\alpha = -\gamma/M(t) \text{ for } (\gamma, \alpha) \text{ in } R\ .$$

This is equivalent to saying

$$\alpha = -\gamma/M(t) \text{ for } |\gamma| \geq 2v_m. \tag{2-30}$$

Recalling that

$$\alpha = 0 \text{ for } |\gamma| < v_m \tag{2-31}$$

it now remains to pick α when $v_m \leq |\gamma| \leq 2v_m$. Assuming that the mean of v is zero, it is best to pick α in R such that α lies as close as possible to the optimum line given by (2-30). The result is

$$\alpha = -2(\gamma - v_m)/M(t) \qquad v_m \leq \gamma \leq 2v_m \tag{2-32}$$

$$\alpha = -2(\gamma + v_m)/M(t) \qquad -2v_m \leq \gamma \leq -v_m\ . \tag{2-33}$$

See Figure 2 for a picture of α as a function of γ as defined by (2-30), (2-31), (2-32) and (2-33).

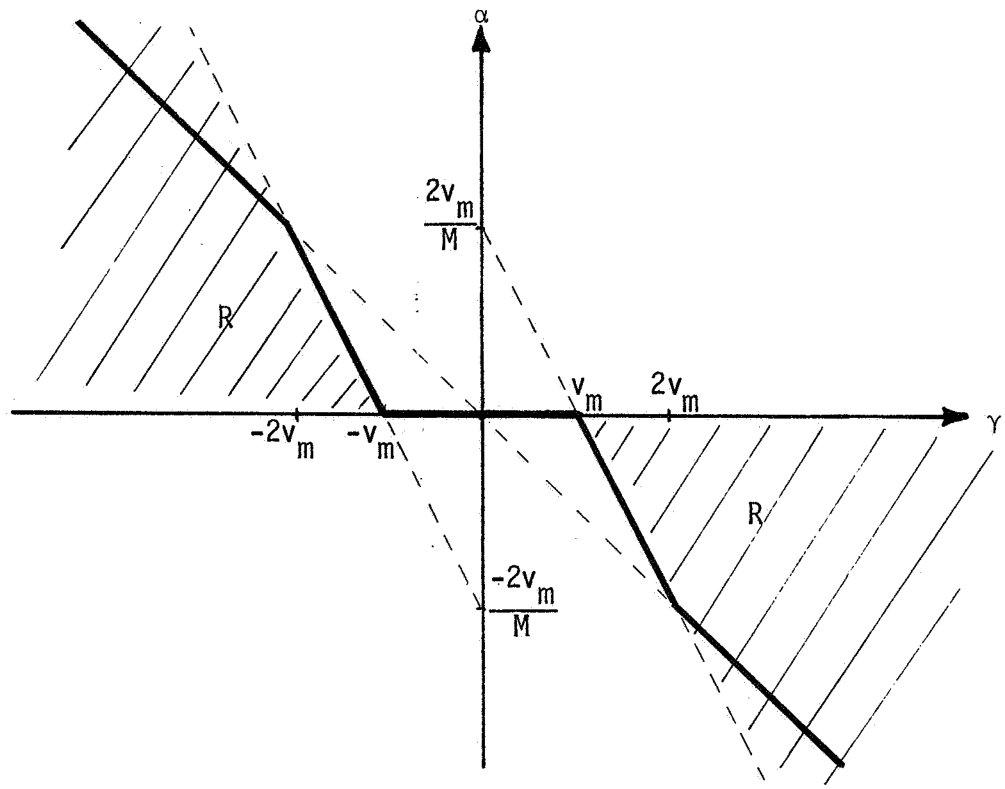


FIGURE 2. α as a Function of γ

The heavy line in Figure 2 represents the best choice of α as a function of γ that can be made with the information available. If the average value of v is known and non-zero, then this should be subtracted from the original measurement y before the equations above are used.

The procedure for obtaining the new estimate $\hat{x}_m$ is as follows:

(a) Calculate γ, i.e., $\gamma = y - h(t, \hat{x}_o)$.

(b) Calculate $M(t) = \sum_{i=1}^{n} h_i^2(t)/b_i$.

(c) Use Figure 2 to get α from γ and M(t).

(d) Use equation (2-19), i.e.,

$$\Delta\tilde{e}_{oi} = (h_i(t)/b_i)\alpha \quad \text{for } i = 1, 2, \ldots, n.$$

(e) Finally use equation (2-12), i.e.,

$$\hat{x}_m = \hat{x}_o + \Delta\tilde{e}_o$$

C. Experimental Results for Extended Estimator

The following examples are intended to illustrate the use of the extended estimator derived in the last section. The example systems and the estimators are simulated on a Control Data 3600 computer using a basic Runge-Kutta integration routine.

Each of these experiments consists of four parts. First, the trajectory of the system is found by solving equation (1-1). Second, the output data from this solution is corrupted with measurement noise as in equation (1-2). Third, these noisy observations are used as inputs to the original estimator outlined in Chapter I to generate $\hat{x}_o$. The cost of this estimate is calculated using equation (2-5). Finally the improved estimate $\hat{x}_m$ is generated using the extended estimator derived in the last section and the cost of this estimate is calculated using equation (2-6). The costs of the two estimators are then compared and graphs are given showing $x(t)$, $\hat{x}_o(t)$, and $\hat{x}_m(t)$ for visual comparison of the estimators.

The model used for the measurements corresponding to equation (1-2) is

$$y(t) = h(t, x) + 0.1 \cdot r_1(t) \cdot |h(t, x)| + 0.1 \cdot r_2(t) \quad (2\text{-}34)$$

where $h(t, x)$ is the variable measured and $y(t)$ is the observed value of this variable after it has been corrupted by measurement noise. For the purposes of the examples which follow, this noise is generated as shown in equation (2-34) where $r_1(t)$ and $r_2(t)$ are two statistically independent random variables which are uniformly distributed from +1 to -1 for each t. This particular form of noise is chosen to demonstrate that the estimation equations are independent of the statistical properties of the noise. The maximum value of this noise, v(t), is assumed known and is calculated using the worst case conditions of equation (2-34). Some knowledge of the bound on h(t, x) is assumed.

Example I

(a) The system trajectory is defined by

$$\dot{x}_1 = x_2 \qquad x_1(0) = 2.0$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 5 \sin t + f_2(t, x) \qquad x_2(0) = 1.0$$

where $f_2(t, x)$ is a time varying nonlinearity which is totally unknown to the estimator. For this example $f_2(t, x)$ is defined as

$$f_2(t, x) = -2e^{-0.1t}x_1^3$$

to specify the system trajectory.

(b) The measurements are defined by

$$y = h(t, x) + v(t)$$

where $h(t, x) = x_1$, v(t) is defined by equation (2-34), and $v_m \doteq 0.25$.

(c) The original estimator as defined by equations (1-19) and (1-20) is

$$\dot{\hat{x}}_{o1} = \hat{x}_{o2} + 2p_{11}(y - \hat{x}_{o1}) \qquad \hat{x}_{o1}(0) = 2$$

$$\dot{\hat{x}}_{o2} = -2\hat{x}_{o1} - 3\hat{x}_{o2} + 5 \sin t + 2p_{21}(y - \hat{x}_{o1}) \qquad \hat{x}_{o2}(0) = 0$$

where $f(t, \hat{x}_o) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is used because $f_2(t, x)$ is unknown to the estimator. Finally

$$\dot{P} = g_{o\hat{x}_o} P + P g^T_{o\hat{x}_o} + 2P[HQ\{y - h(t, \hat{x}_o)\}]_{\hat{x}_o} P + \frac{1}{2} W^{-1}$$

where:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad g_{o\hat{x}_o} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Q = [1] \qquad W = \frac{1}{2} I$$

$$[HQ\{y - h(t, \hat{x}_o)\}]_{\hat{x}_o} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

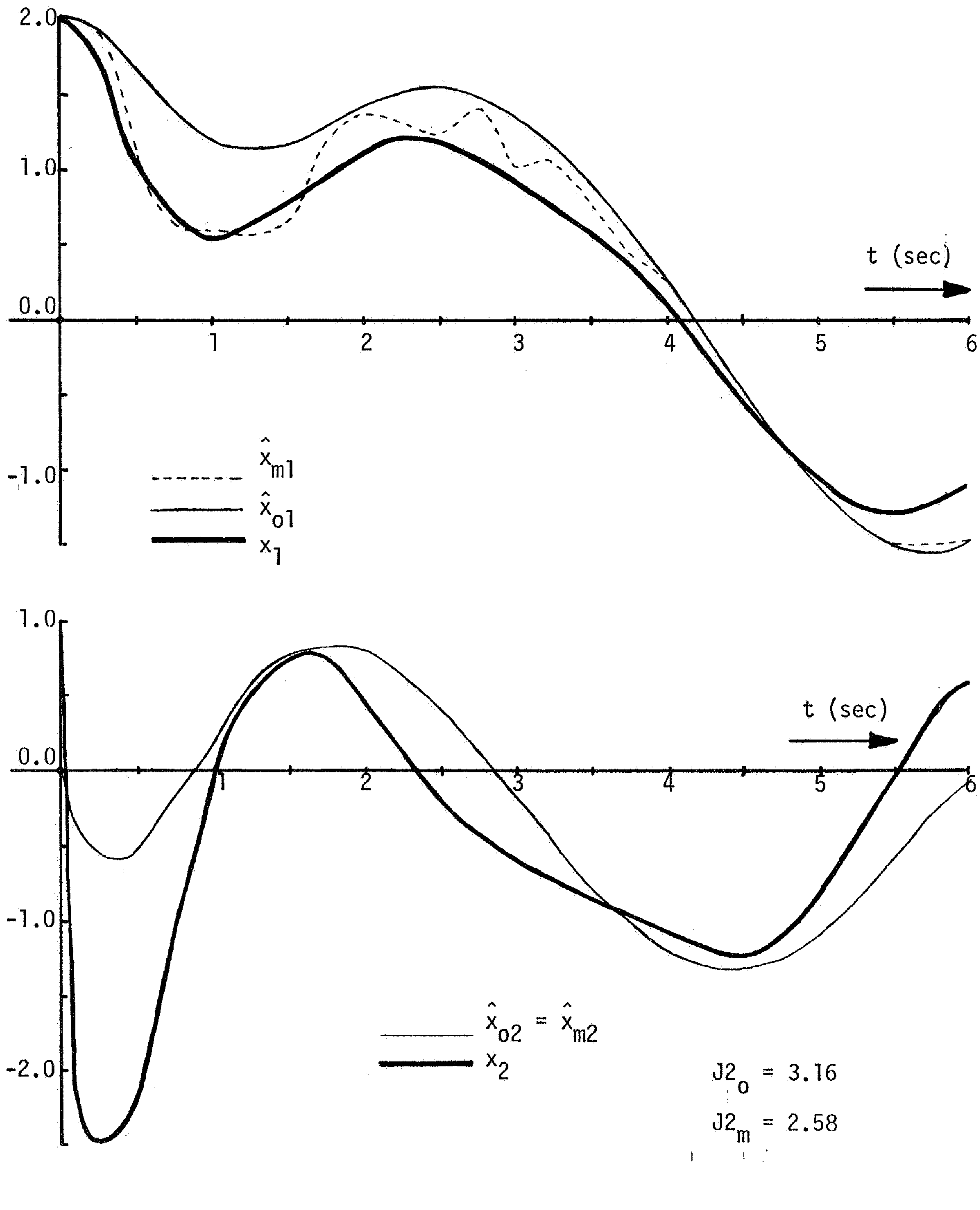
Since the system is linear the above equation for P may be solved for steady state by setting $\dot{P} \equiv 0$ and the result is

$$p_{11} = 0.549266$$
$$p_{12} = p_{21} = -0.198307$$
$$p_{22} = 0.285763$$

The estimates generated by the above equations correspond to $\hat{x}_o$ and the cost of this estimate is found using (2-5) and (2-6) with B taken as the identity matrix.

(d) Finally, $\hat{x}_m$ is generated using the procedure outlined in the last section and the cost of this estimate is found using (2-7) and (2-8) with B again taken as the identity matrix.

With $t_f = 6$ seconds the results of this experiment are:



Graph I. Example I

$J2_o = 3.156$ - original estimates

$J2_m = 2.583$ - improved estimates

$J2_m - J2_o = -0.577 \leq 0$

This is about an 18% improvement in estimation with respect to the given cost function and $t_f = 6$ seconds. See Graph I for a plot of these estimates.

Example II

(a) The system trajectory is the same as that of Example I.

(b) The measurements are defined as before by

$$y = h(t, x) + v(t)$$

where $h(t, x) = \frac{1}{2} x_1 - x_2$, $v(t)$ is defined by equation (2-34), and an estimate of v_m gives $v_m \doteq 0.4$.

(c) The original estimator equations are derived from the system equations using (1-19) and (1-20), but in this example the system's time varying nonlinearity, i.e., $-2e^{-0.1t}x_1^3$, is assumed known. Consequently, this term affects the estimator and P equations. The system is non-linear so that the P equations have no steady state solution. The time varying solution must then be used. The initial condition on the P equations is taken as

$$P(0) = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The original estimator equations are

$$\dot{\hat{x}}_{o1} = \hat{x}_{o2} + (p_{11} - 2p_{12})(y - \frac{1}{2}\hat{x}_{o1} + \hat{x}_{o2})$$

$$\dot{\hat{x}}_{o2} = -2\hat{x}_{o1} - 3\hat{x}_{o2} + 5 \sin t - 2e^{-0.1t}\hat{x}_{o1}^3 +$$

$$+ (p_{12} - 2p_{22})(y - \frac{1}{2}\hat{x}_{o1} + \hat{x}_{o2})$$

$$\dot{p}_{11} = 1 + 2p_{12} - \frac{1}{2}(p_{11} - 2p_{12})^2$$

$$\dot{p}_{12} = p_{22} - 3p_{12} - 2p_{11}(1 + 3e^{-0.1t}\hat{x}_{o1}^2) +$$

$$+ p_{12}(p_{12} - \frac{1}{2}p_{11} - 2p_{22}) + p_{11}p_{22}$$

$$\dot{p}_{22} = 1 - 6p_{22} - 4p_{12}(1 + 3e^{-0.1t}\hat{x}_{o1}^2) - \frac{1}{2}(p_{11} - 2p_{22})^2$$

with $p_{12} = p_{21}$.

As before the cost of this estimate is calculated, but for demonstration purposes the weighing matrix B may be defined as

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

(d) The improved estimate $\hat{x}_m$ is generated and its cost is found using B above.

Two sets of results are obtained for this problem. First, if the initial conditions on $\hat{x}_o$ are chosen as

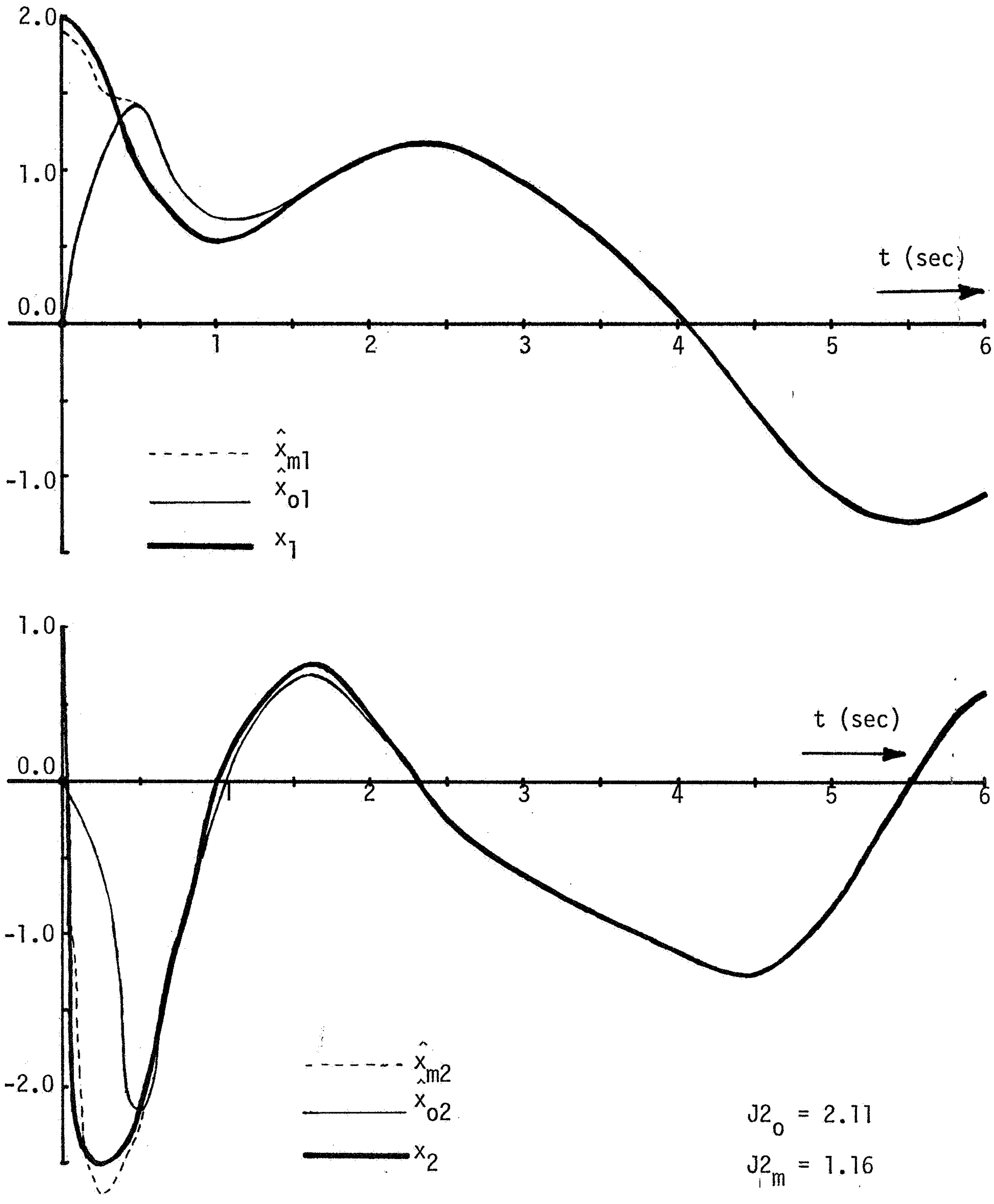
$$\hat{x}_{o1}(0) = 0 \qquad \hat{x}_{o2}(0) = 0$$

the result for $t_f = 6$ seconds is

$J2_o = 2.117$

$J2_m = 1.159$

% improvement $\doteq$ 50%



Graph II. Example II

This result is plotted in Graph II. In the second case, the initial conditions on $\hat{x}_o$ are chosen as:

$$\hat{x}_{o1}(0) = 2 \qquad \hat{x}_{o2}(0) = 0$$

The result for $t_f = 6$ seconds is:

$$J2_o = 0.074$$

$$J2_m = 0.047$$

$$\% \text{ improvement} \doteq 30\%$$

This result is not plotted because the error in estimation is too small to be seen clearly.

It should be noted that the large improvement in both estimates for this case is due to starting the estimate of $\hat{x}_{o1}$ at the true value of $x_1(0)$, i.e.,

$$\hat{x}_{o1}(0) = x_1(0) .$$

<u>Example III</u>

(a) The system trajectory is defined by

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0.5$$

$$\dot{x}_2 = -(1 - \Delta)x_1 \qquad x_2(0) = 0.0$$

where Δ is a constant but unknown parameter variation. For the purpose of this example Δ is set at 0.25, but this value does not affect the estimator in any way because it is assumed unknown.

(b) The measurements are defined by

$$y = h(t, x) + v(t)$$

where $h(t, x) = a(t)x_1 + x_2$ for $a(t) = 2 + \sin(0.5\,t)$. $v(t)$ is a random variable with a uniform distribution from +0.1 to -0.1. Hence, $v_m = 0.1$.

(c) The original estimator as defined by equations (1-26) and (1-27) is:

$$\dot{\hat{x}}_{o1} = \hat{x}_{o2} + 2[a(t)p_{11} + p_{12}][y - (a(t)\hat{x}_{o1} + \hat{x}_{o2})] \qquad \hat{x}_{o1}(0) = 0$$

$$\dot{\hat{x}}_{o2} = -\hat{x}_{o1} + 2[a(t)p_{21} + p_{22}][y - (a(t)\hat{x}_{o1} + \hat{x}_{o2})] \qquad \hat{x}_{o2}(0) = 0$$

$$p_{12} = p_{21}$$

$$\dot{p}_{11} = 1 + 2p_{12} - 2(a(t)p_{11} + p_{12})^2 \qquad p_{11}(0) = 0$$

$$\dot{p}_{12} = p_{22} - p_{11} - 2(a(t)p_{11} + p_{12})(a(t)p_{12} + p_{22}) \qquad p_{12}(0) = 0$$

$$\dot{p}_{22} = 1 - 2p_{12} - 2(a(t)p_{12} + p_{22})^2 \qquad p_{22}(0) = 0$$

The derivation of the above used the following:

$$g_o(t, \hat{x}_o) = \begin{bmatrix} \hat{x}_{o2} \\ -\hat{x}_{o1} \end{bmatrix} \qquad \Delta g(t, \hat{x}_o) = \begin{bmatrix} 0 \\ \Delta \cdot \hat{x}_{o1} \end{bmatrix}$$

$$k(t, \hat{x}_o)u(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Q = [1] \qquad W = \frac{1}{2} I$$

$$H = \begin{bmatrix} a(t) \\ 1 \end{bmatrix} \qquad g_{o\hat{x}_o} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since Δ is unknown to the estimator

$$f(t, \hat{x}_o) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The cost of this estimate is defined by (2-5) and (2-7) with

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} .$$

(d) $\hat{x}_m$ is generated using the procedure outlined in Chapter II, Section B, and the cost of this estimate is found using the given B matrix.

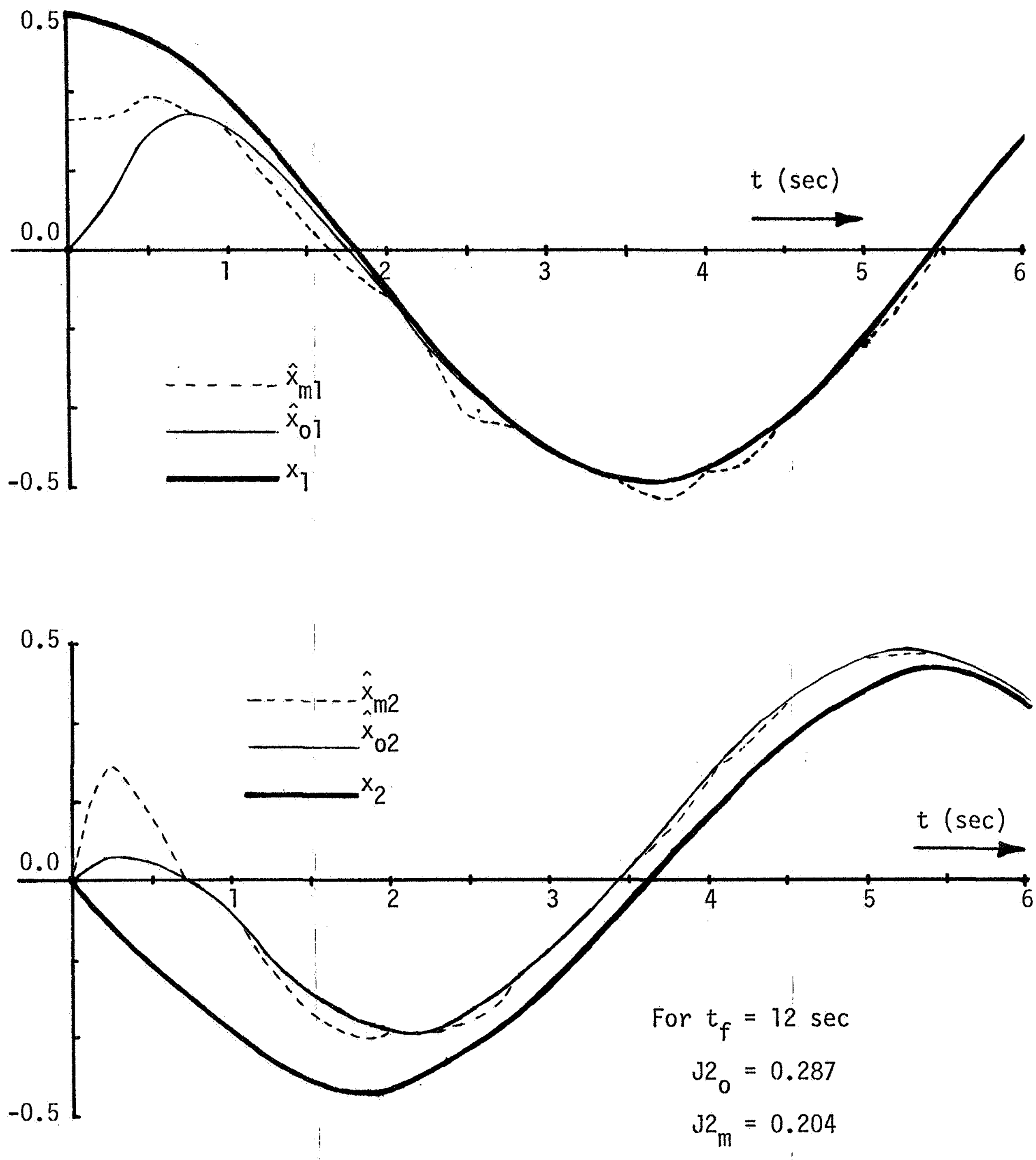
For a final time of 12 seconds the results are:

$J2_o = 0.287$

$J2_m = 0.204$

% improvement $\doteq$ 29%

See Graph III for a plot of these results. Inspection of this graph reveals that at certain times, i.e., t = 3.75 seconds, the improved estimate appears to be worse than the original estimate. However, calculation of the instantaneous cost or the integrands of the cost functions $J2_o$ and $J2_m$ at any such time relative to the given B matrix demonstrates that $\hat{x}_m$ is more accurate than $\hat{x}_o$. The difficulty lies in the fact that some components of $\hat{x}_m$ will lose accuracy if others are improved. The cost function $J2_m$ however is a function of all components of $\hat{x}_m$ and it is the overall cost that is less, not the cost relative to any single component of $\hat{x}_m$.



Graph III. Example III

D. Comments on the Extended Estimator

The extended estimator improves estimation in all of the examples tested. The largest improvement occurs when the original estimator is doing a poor job, and the improvement is negligible if the original estimator is estimating accurately. The main advantage of this technique is to improve estimation during the initial period when the estimator is converging on the correct estimates or after some disturbance to the system which causes a large jump in the estimation error. The technique is relatively simple to implement once the original estimator has been constructed, and it has little effect on the estimation if the estimation is accurate.

CHAPTER III

SIMPLIFIED ESTIMATOR

A. Problem Definition

The problem discussed in this chapter is the practical application of the estimation technique summarized in Chapter I, Section B. It is desired that a way be found to reduce the large amount of computation necessary for this estimator so that it may be implemented in a more practical manner. It is shown that for some non-linear and/or time varying systems it is possible to "linearize" the estimator in such a way that adequate if not optimal estimation occurs. First the non-linear and time varying terms are removed from the system and the matrix Riccati equation is solved for steady state using the remaining linear system. Then these steady state values are used in the estimation equations derived from the full non-linear and/or time varying system. This greatly reduces the computation required by the estimator because the matrix Riccati equation need only be solved once. It is also convenient that this solution is run "off-line."

B. Theoretical Development of the Simplified Estimator

Using equations (1-1), (1-2) and (1-9) the system equations can be written as:

$$\dot{x}(t) = g_0(t, x) + f(t, x) \tag{3-1}$$

$$y(t) = h(t, x) + v(t) \tag{3-2}$$

The original estimation equations (1-26) and (1-27) for this system are:

$$\dot{\hat{x}} = g_o(t, \hat{x}) + 2P(t)H(t, \hat{x})Q\{y(t) - h(t, \hat{x})\} + f(t, \hat{x})$$

$$\dot{P} = g_{o\hat{x}}(t, \hat{x})P + Pg^T_{o\hat{x}}(t, \hat{x}) + 2P[HQ\{y - h(t, \hat{x})\}]_{\hat{x}}P + \frac{1}{2} W^{-1}$$

The use of these equations requires that the $f(t, \hat{x})$ term be set to zero since $f(t, x)$ is the unknown part of the system. Hence, the actual, but unknown, value of $f(t, x)$ affects the above equations only indirectly through the measurements of the system output, i.e., $y(t)$. Likewise, the form of $f(t, x)$ has no effect on the form of the final estimator equations because $f(t, x)$ is assumed totally unknown.

The appearance of $f(t, \hat{x})$ in the estimator suggests that it might be possible to use this term to simplify the equations. This is done by breaking up $g_o(t, x)$ into two parts. Let the first part contain all of the linear terms of $g_o(t, x)$ and the second part contain all of the non-linear and time varying terms. Let

$$g_o(t, x) = Ax + s(t, x) \tag{3-3}$$

where A is an $n \times n$ constant matrix and $s(t, x)$ is a non-linear and/or time varying n-vector. Now write the system equation (3-1) as

$$\dot{x} = Ax + f'(t, x) \tag{3-4}$$

where

$$f'(t, x) = s(t, x) + f(t, x)\,. \tag{3-5}$$

The equations of the simplified estimator are generated from (1-26) and

(1-27) with Ax in the role of $g_0(t, x)$ and $f'(t, x)$ in the role of $f(t, x)$ even though $f'(t, x)$ is not completely unknown as assumed in the derivation of the original estimator.

Originally the $f(t, \hat{x})$ term of the estimator was set to zero because it was unknown, but in the simplified estimator this term becomes $f'(t, \hat{x})$ which is partially known. In this estimator $f(t, \hat{x})$ is again set to zero which gives:

$$f'(t, \hat{x}) = s(t, \hat{x}) \tag{3-6}$$

The estimator equations with all of the above changes become the equations of the simplified estimator. These equations are

$$\dot{\hat{x}} = A\hat{x} + 2PH(t, \hat{x})Q\{y(t) - h(t, \hat{x})\} + s(t, \hat{x}) \tag{3-7}$$

$$\dot{P} = AP + PA^T + 2P[HQ\{y - h(t, \hat{x})\}]_{\hat{x}}P + \frac{1}{2} W^{-1} \tag{3-8}$$

This estimator is much simpler than the original estimator because the P equation may be solved for a steady state solution. This solution may be found beforehand and the constant P which results is denoted by P_{ss}, i.e., P steady state. The use of P_{ss} greatly reduces the amount of further computation required by the estimator. The existence of this steady state solution for P is a consequence of the fact that the differential equation for P depends only upon the linear part of the system. P_{ss} is found by setting $\dot{P} \equiv 0$ in the P equation, (3-8), and solving the resulting equation for P_{ss}. The simplified estimator is now:

$$\dot{\hat{x}} = A\hat{x} + 2P_{ss}H(t, \hat{x})Q\{y(t) - h(t, \hat{x})\} + s(t, \hat{x}) \tag{3-9}$$

It should be noted that the optimization of this estimator is carried out assuming that $s(t, x)$ is completely unknown. The simplified estimator, therefore, cannot be optimum because the optimization was carried out partially ignoring the form of the non-linear and time varying terms in the system. This is the price that must be paid for the simplification that is achieved.

There are two major theoretical drawbacks to this simplification technique. First, if the non-linear and time varying terms of the system play a dominant role in the response of the system, the estimation will probably be very poor because the form of these terms is partially ignored in the optimization process. Second, the solution of $\dot{P} = 0$ for P_{ss} may lead to difficulty. This solution depends on the A matrix and on $h(t, x)$, the function which relates the measurements to the states of the system. Since this A matrix represents only part of the true system it may not be possible to find an adequate solution for P_{ss}. A and $h(t, x)$ may be such that no steady state solution for P_{ss} exists. This happens if A is not controllable or if $h(t, x)$ is time varying. Estimation in this case is either impossible or nonsense. Consequently, the applicability of this technique depends entirely on the problem at hand.

C. <u>Experimental Results for the Simplified Estimator</u>

In this section an example is given which demonstrates the advantages of the simplified estimator for a case where it is applicable. Any particular problem must be examined carefully to see if this technique can be used.

(a) The system is defined by:

$$\dot{x}_1 = x_2 \qquad\qquad x_1(0) = 2.0$$

$$\dot{x}_2 = -2x_1 - 3x_2 + 5 \sin (t) - 2e^{-0.1t}x_1^3 \qquad x_2(0) = 1.0$$

(b) The measurements are defined by

$$y = x_1 + v$$

where $v = 0.1 \cdot |x_1| \cdot r_1 + 0.1 \cdot r_2$. r_1 and r_2 are independent random variables, uniformly distributed from +1 to -1. This results in $v_m \doteq 0.25$.

(c) The full scale estimator as defined by equations (1-26) and (1-27) is

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2p_{11}(y - \hat{x}_1)$$

$$\dot{\hat{x}}_2 = -2\hat{x}_1 - 3\hat{x}_2 + 5 \sin t - 2e^{-t/10}\hat{x}_1^3 + 2p_{21}(y - \hat{x}_1)$$

$$\hat{x}_1(0) = 0 \qquad\qquad \hat{x}_2(0) = 0$$

$$p_{12} = p_{21}$$

$$\dot{p}_{11} = 1 + 2(p_{12} - p_{11}^2)$$

$$\dot{p}_{12} = p_{22} - 3p_{12} - 2p_{11}(1 + 3e^{-t/10}x_1^2 + p_{12})$$

$$\dot{p}_{22} = 1 - 2p_{12}^2 - 6p_{22} - 4p_{12}(1 + 3e^{-t/10}x_1^2)$$

$$p_{11}(0) = 3 \qquad p_{12}(0) = 1 \qquad p_{22}(0) = 3$$

This estimate is plotted in Graph IV.

(d) A possible technique of reducing the amount of computation required by the full estimator above would be to completely ignore the non-linear term

$$-2e^{-t/10}x_1^3$$

which appears in the original system equations. If this is done, the remaining system is linear and a simpler set of estimation equations is derived

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2p_{11}(y - \hat{x}_1) \qquad \hat{x}_1(0) = 0$$

$$\dot{\hat{x}}_2 = -2\hat{x}_1 - 3\hat{x}_2 + 5\sin(t) + 2p_{21}(y - \hat{x}_1) \qquad \hat{x}_2(0) = 0$$

with the steady state solution of the P equations:

$$p_{11} = 0.549266$$

$$p_{12} = p_{21} = -0.198307$$

$$p_{22} = 0.285763$$

This estimate is plotted in Graph V, and comparing this with Graph IV it is obvious that this method of simplifying the estimator gives rather poor results.

(c) The simplified estimator described earlier in this chapter is now tested and compared with the results of the two estimators above. The important equations are

$$g_o(t, x) = Ax + s(t, x) = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 - 2e^{-t/10}x_1^3 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad s(t, x) = \begin{bmatrix} 0 \\ -2e^{-t/10}x_1^3 \end{bmatrix} .$$

Then

$$f'(t, \hat{x}) = s(t, \hat{x}).$$

Using equations (3-7) and (3-8) and solving for the steady state P matrix the estimation equations become:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2p_{11}(y - \hat{x}_1) \qquad \hat{x}_1(0) = 0$$

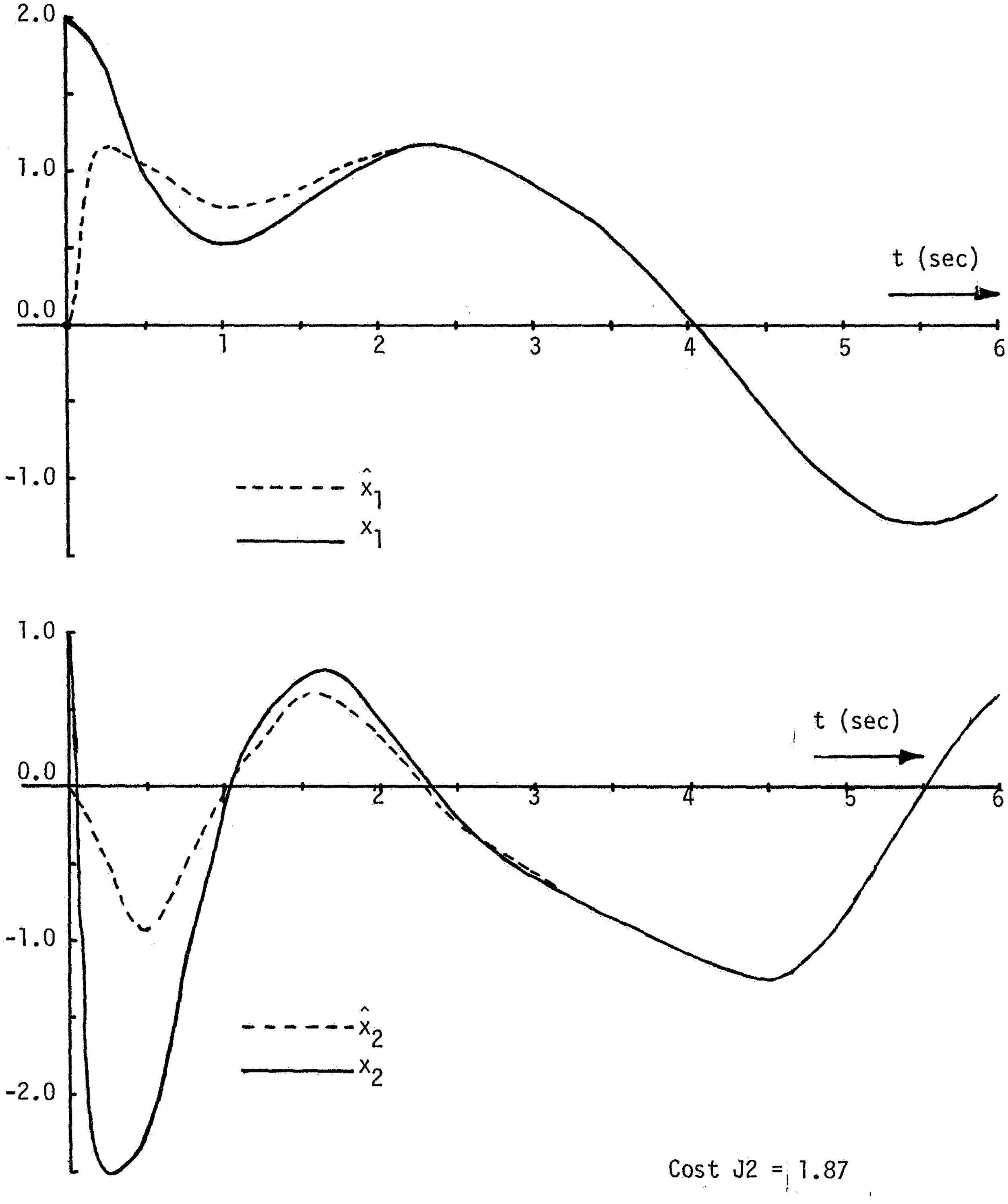
$$\dot{\hat{x}}_2 = -2\hat{x}_1 - 3\hat{x}_2 + 5 \sin(t) + 2p_{21}(y - \hat{x}_1) - 2e^{-t/10}x_1^3 \qquad \hat{x}_2(0) = 0$$

$$p_{11} = 0.549266$$

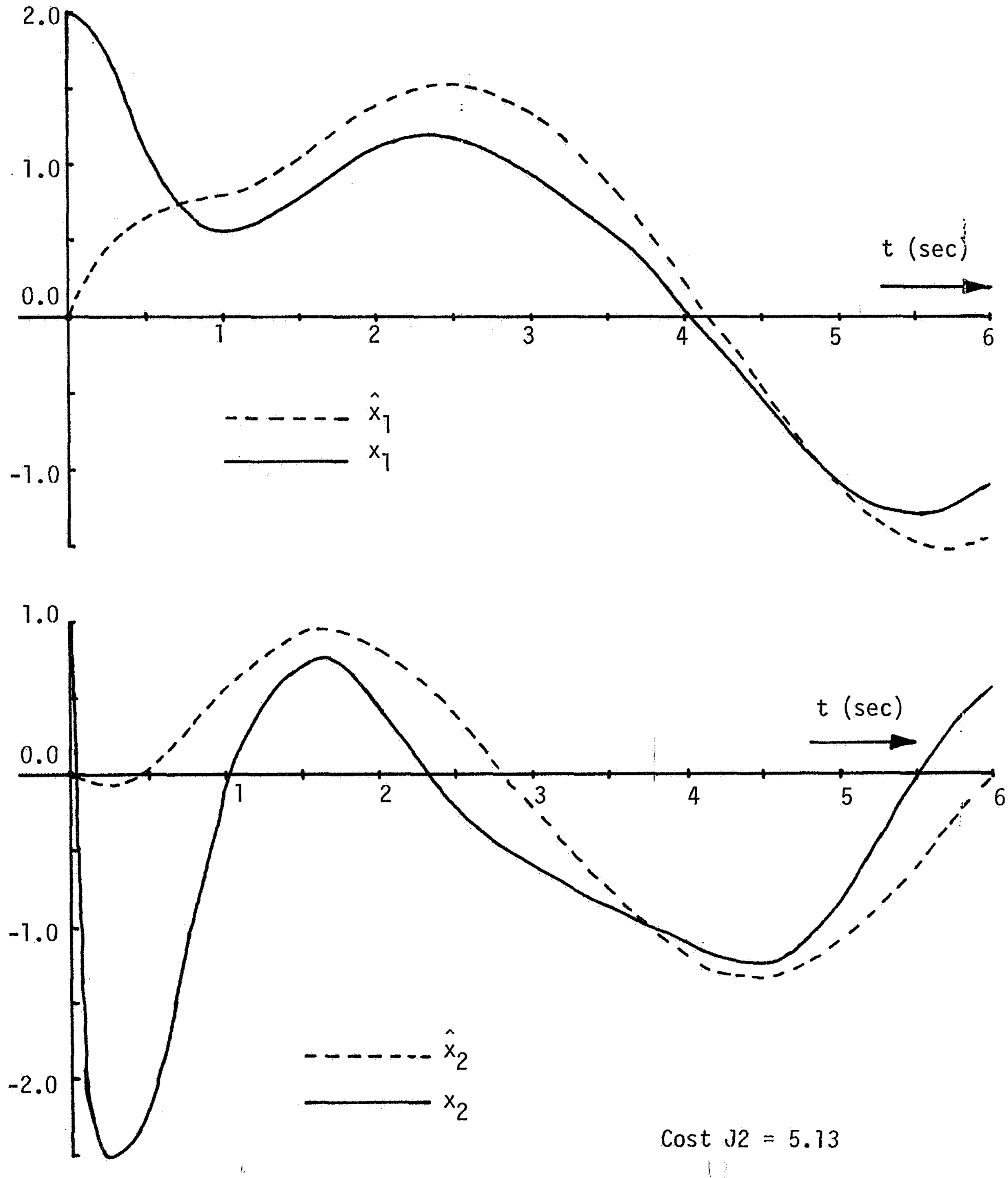
$$p_{21} = -0.198307$$

Notice that this estimator is exactly the same as the one derived in b above except the non-linear time varying term appears as an input. This means that the amount of computation required by the two methods is practically the same, which in both cases is much less than that of the full scale estimator. The trajectory of this estimator is plotted in Graph VI and the estimation is obviously much better than that of the other simplified estimator. It is not as good as the full scale estimator, however. This loss in estimation accuracy was expected. The cost of these three estimates was calculated using (2-1) with B equal to the identity matrix and t_f = 6 seconds. The results are:

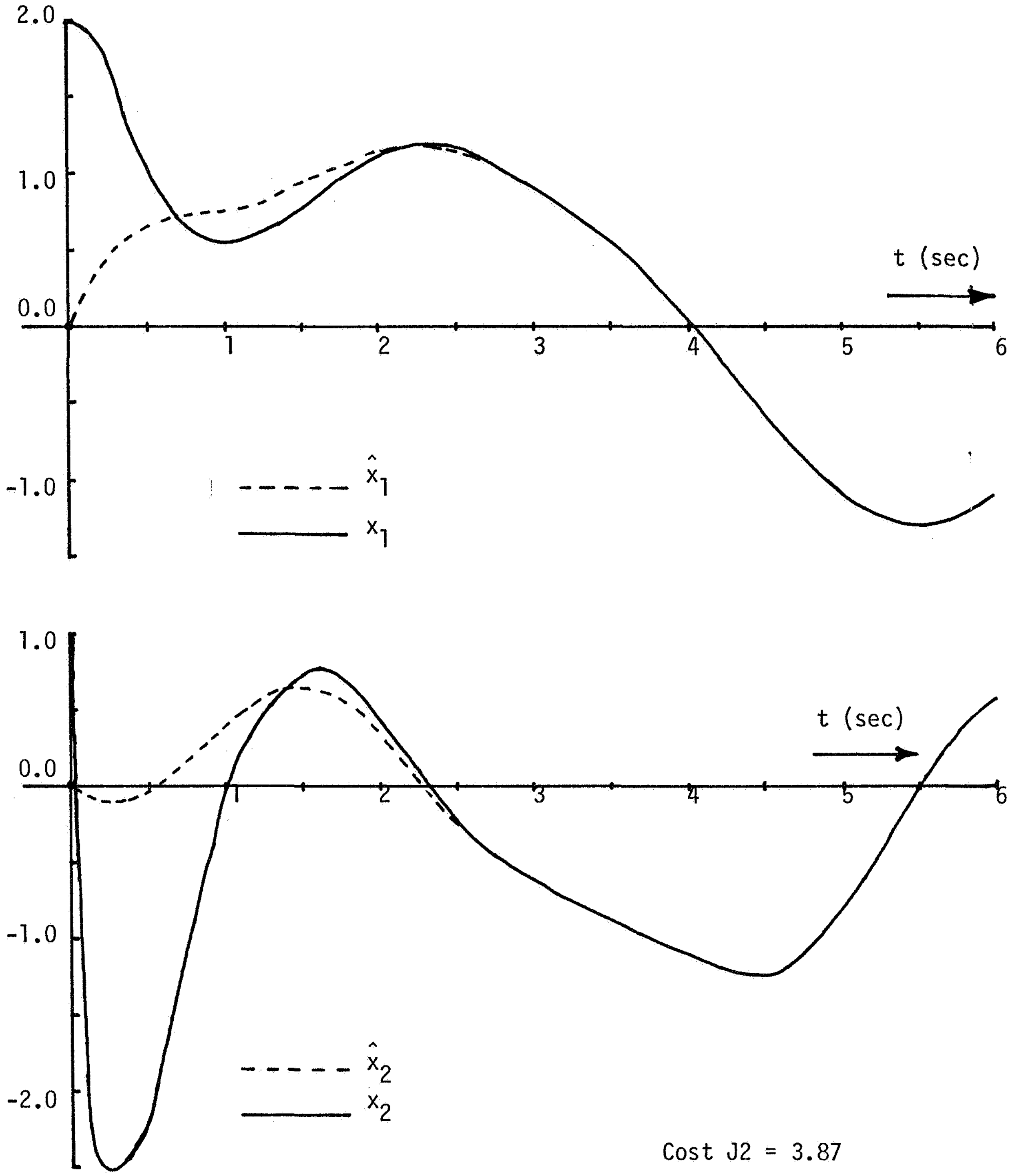
	Cost J2	Computer Time (sec)
Full Estimator	1.87	134
Linear System Estimator	5.13	100
Simplified Estimator	3.87	101



Graph IV. Full Estimator



Graph V. Linear System Estimator



Graph VI. Simplified Estimator

D. Comments on the Simplified Estimator

The advantage of the simplified estimator is the reduction in computation required in estimation. This reduction becomes much greater as the order of the system to be estimated increases. The full estimator requires that the P equation, i.e., the matrix Riccati equation, be solved each time the estimator is used. This equation is a non-linear time varying matrix differential equation and its solution amounts to solving $n(n + 1)/2$ first order non-linear time varying interdependent differential equations because it is found to be a symmetric matrix differential equation. This is an excessive amount of computation for any high order system. The simplified estimator, however, insures that the P equation has a steady state solution which must only be solved for once. This solution is run beforehand and the steady state values of P are used in the estimator. Now, when the estimator is used only n differential equations must be solved rather than $n + n(n + 1)/2$ equations. This is very convenient assuming that the loss in estimation accuracy can be accepted in exchange for the reduction in computation. This loss in estimation accuracy is a function of the system that is to be estimated and will vary greatly from system to system.

This technique depends on splitting up the system into a linear and a non-linear part, and the principal part of the estimator depends on the linear part of the system. The major problem is that many systems may not have a linear part. Those systems which do have linear terms still may not have enough linear terms around which a reasonable estimator may be derived. If the A matrix which results from the linear part of the system is not controllable then it cannot be assured that a steady state solution for the P matrix exists. If $h(x, t)$ is time varying then no steady state solution exists. If the nonlinear terms in the system are the major terms which determine the response of the system, then the estimation may be very poor because these terms were not considered in the optimization of the estimator. However, if none of the above occurs in a particular problem, then the simplified estimator may be used to great advantage as demonstrated.

CHAPTER IV

CONCLUSIONS AND SUGGESTIONS

This report describes two techniques designed to make the sequential estimator summarized in Chapter I more useful. The first suggests a way of improving the accuracy of the estimator given knowledge of the bounds on the measurement noise. The second suggests a way of reducing the amount of computation involved in estimation by giving up some accuracy. Both techniques have definite limitations and their use must be determined by a careful analysis of the problem at hand.

This report is an attempt to improve upon a basic sequential estimation technique. Further work along this line might lead to more methods of improvement, but it seems that any major improvement would have to come from a more basic analysis of the whole estimation problem rather than from trying to improve upon an estimator whose basic design has already been determined.

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